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INVARIANT SUBMODELS OF A SPECIAL VORTEX†

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Two invariant submodels of a spherically partially invariant model of gas dynamics called a special vortex are investigated: a steady-state model and a homogeneous submodel. A complete analytical description of them is given: all the invariant functions specifying the solution have a representation in terms of an auxiliary function and its derivatives. This function is the solution of the first-order ordinary differential equation for the steady-state special vortex and the Schwarz equation for the homogeneous special vortex. A qualitative description of the gas in the homogeneous special vortex is given. The characteristic feature of this motion is the formation of a gas cloud from the rarefied medium accompanying its motion towards the observer and the subsequent dispersion again to a rarefied state (in the limit to a vacuum) at infinity. A barochronic homogeneous special vortex is fully described. It is proved that the special vortex is generated by special initial data: the algebraic invariants of the Jacobian matrices of the velocity vector field depend solely on the invariant independent variables of time and the radial coordinate. A representation of the invariants in terms of the parameters of the singular vortex is obtained. © 2003 Elsevier Ltd. All rights reserved.

Exact solutions associated with the group of rotations $SO(3)$ are interest by virtue of its position among the other subgroups of the Galilean group, which is the fundamental symmetry group of mathematical models in continuum mechanics. Spherically symmetric solutions in gas dynamics and hydrodynamics are classical and have numerous applications. Ovsyannikov [1] discovered the "special vortex", a solution of the equations of hydrodynamics which is partially invariant with respect to the $SO(3)$ group and for which the radial component of the velocity vector is spherically symmetric, but the component of the velocity vector which is tangent to the spheres is non-zero. The name of this solution is explained by the fact that a vortex having, in spherical coordinates, a zero radial component everywhere on a sphere, apart from its poles at which it becomes infinite, corresponds to the special initial distribution of its velocity field. The term "special vortex", by virtue of its brevity and expressiveness, is also transferred to the entire class of such solutions, which more accurately and lengthily can be called spherically partially invariant (symmetric) or $SO(3)$ partially invariant solutions. This is a broad class of physically interesting exact solutions. The kinematics and dynamics of the gas motions corresponding to them are quite complex. On account of this, it is of interest to investigate the exact solutions of this submodel, in particular, the invariants, for a more detailed description of the motion.

A general description of a special vortex is given in Section 1: the reduction of the equations of gas dynamics to invariant and overdetermined systems and the adduction of the latter into an involution (the derivation of the compatibility conditions). The overdetermined system is integrated in final form in the solutions of the invariant system.

The property of the Jacobian matrix of the velocity vector field of a special vortex of having algebraic invariants and eigenvalues, which depend solely on invariant independent variables, is proved in Section 2. This gives an invariant characteristic of the solution, which is independent of the formulae of the representation. This property is general for a wide class of regular partially invariant solutions.

A steady-state special vortex is described in Section 3. The invariant system for this vortex reduces to a first-order ordinary differential equation and finite relations which express all of the required functions in terms of its solution. A steady-state vortex is defined outside a sphere of radius $r_* > 0$.

A homogeneous special vortex for a polytropic gas is investigated in Sections 4 and 5. In this case, the invariant subsystem reduces to an inhomogeneous Schwarz equation for the auxiliary function h. All of the required functions are expressed in terms of it and its derivatives. For this solution, the invariants of the Jacobi matrix of the velocity field and the eigenvalues depend solely on time.

It is proved in Section 6 that, in the case of an adiabatic exponent $\gamma = 5/3$, the Schwarz equation, which is of the third order, decomposes into a non-linear first-order equation and a second-order linear equation, the solution of which determines the right-hand side of the non-linear equation. This result, which is of a general nature, serves as a basis for studying the qualitative properties of the gas motion

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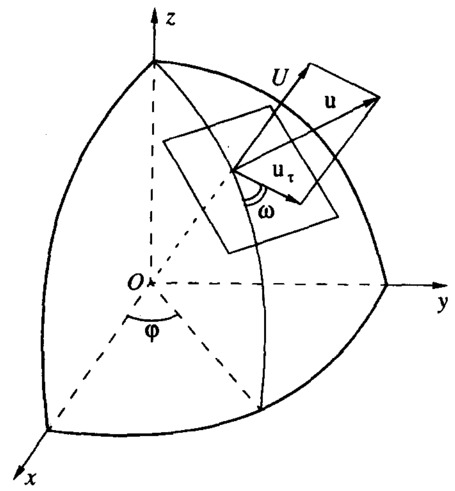

Fig. 1

in a homogeneous special vortex, which are described in Sections 7 and 8. A stage in which there is a condensation of a gaseous formation (a cloud), formed from the rarefied medium as it approaches an observer, and a subsequent stage of dispersion and rarefaction, are characteristic for the gas motion in a special vortex. Both finite and infinite intervals of existence of the solution are possible.

A barochronic homogeneous special vortex is investigated in Section 8. A description of the trajectories of the gas particles as rectilinear generatrices of the three-dimensional unparted hyperboloid in the space of the events $\mathbb{R}^4$ $(t, \mathbf{x})$ and the collapse manifold is given for this vortex.

1. THE SUBMODEL OF A SPECIAL VORTEX [1]

Spherical coordinates (r, θ, φ) using the formulae

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \tag{1.1}$$

and the components of the velocity vector (U, V, W) are introduced in the space $\mathbb{R}^3(\mathbf{x})$ together with the Cartesian coordinates $\mathbf{x} = (x, y, z)$ and the corresponding components of the velocity vector $\mathbf{u} = (u, v, w)$ [2]. On the spheres $r = \text{const}$, the radial component of the velocity U is equal to the magnitude of the normal component of the velocity vector, and the vector $\mathbf{u}_\tau = (V, W)$ is its tangential component. In a plane, which is tangential to the spheres, it is convenient to introduce the modulus $\mathbf{u}_\tau$: $H = \sqrt{(V^2 + W^2)}$ and the angle ω of its deviation from the meridian (Fig. 1)

$$V = H\cos\omega, \quad W = H\sin\omega \tag{1.2}$$

In the coordinates (1.1) and (1.2), the equations of gas dynamics have the form

$$\begin{aligned}
&DU + \rho^{-1}p_r = r^{-1}H^2 \\
&DH^2 + 2(\rho r)^{-1}H(\cos\omega p_\theta + (\sin\theta)^{-1}\sin\omega p_\varphi) = -2r^{-1}UH^2 \\
&D\omega + (\rho r)^{-1}((\sin\theta)^{-1}\cos\omega p_\varphi - \sin\omega p_\theta) = -r^{-1}H\operatorname{ctg}\theta\sin\omega \\
&D\rho + \rho\operatorname{div}\mathbf{u} = 0, \quad DS = 0, \quad p = f(\rho, S)
\end{aligned} \tag{1.3}$$

where ρ is the density, p is the pressure and S is the entropy of the gas. The function f specifies the equation of state of the gas. The operators D and div have the form

$$\begin{aligned}
&D = \partial_t + U\partial_r + r^{-1}H(\cos\omega\partial_\theta + (\sin\theta)^{-1}\sin\omega\partial_\varphi) \\
&\operatorname{div}\mathbf{u} = r^{-2}(r^2U)_r + (r\sin\theta)^{-1}((H\cos\omega\cos\theta)_\theta + (H\sin\omega)_\varphi)
\end{aligned} \tag{1.4}$$

Equations (1.3) for an equation of state of general form allow of the Lie algebra L_{11} (Galilean algebra, expanded by a uniform extension of the variables $(t, \mathbf{x})$). The $SO(3)$ algebra, which corresponds to the group of rotations $SO(3)$, is a subalgebra of L_{11} and, in the coordinates (1.1) and (1.2), has the basis

$$\begin{aligned} X &= -\sin\varphi\partial_\theta - \cos\varphi\,\mathrm{ctg}\,\theta\partial_\varphi + (\sin\theta)^{-1}\cos\varphi\partial_\omega \\ Y &= \cos\varphi\partial_\theta - \sin\varphi\,\mathrm{ctg}\,\theta\partial_\varphi + (\sin\theta)^{-1}\sin\varphi\partial_\omega \\ Z &= \partial_\varphi \end{aligned} \tag{1.5}$$

In the basis space of the variables $(t, r, \theta, \varphi, U, H, \omega, \rho, S)$, the $SO(3)$ algebra with the operators (1.5) has the invariants t, r, U, H, ρ and S. The quantity ω is an extraneous function. Hence, the $SO(3)$ algebra with the operators (1.5) generates regular partially invariant solutions of Eqs (1.3) of rank 2 and defect 1. In the case of these solutions, the invariant functions U, H, ρ and S depend solely on the invariant independent variables t, r, θ, φ. The representation of the required regular, partially invariant solutions has the form

$$U = U(t, r), \quad H = H(t, r), \quad \rho = \rho(t, r), \quad S = S(t, r), \quad \omega = \omega(t, r, \theta, \varphi) \tag{1.6}$$

After substituting representation (1.6) into system (1.3), factor equations are obtained which, according to the general theory [3], decompose into an invariant subsystem

$$D_0U + \rho^{-1}p_r = r^{-1}H^2, \quad D_0(rH) = 0, \quad D_0S = 0, \quad p = f(\rho, S) \tag{1.7}$$

where $D_0 = \partial_t + U\partial_r$ and an overdetermined system for the function ω

$$\begin{aligned} k\sin\theta D_0\omega + \sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi &= -\cos\theta\sin\omega \\ \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi &= \cos\theta\cos\omega + h\sin\theta \end{aligned} \tag{1.8}$$

The auxiliary invariant functions

$$k = r/H, \quad h = k(\rho^{-1}D_0\rho + r^{-2}(r^2U)_r) \tag{1.9}$$

have been introduced here. It is assumed that $H \neq 0$. If $H = 0$, then by virtue of relations (1.2), the tangential component $\mathbf{u}_\tau$ of the velocity vector is equal to zero. In this case, system (1.3) describes the well-known spherically-symmetric motions of a gas.

To overdetermined system (1.8) is compatible by virtue of the equation

$$kD_0h = h^2 + 1 \tag{1.10}$$

Equation (1.9) and (1.10) supplement the invariant system (1.7) to form a closed system in the functions U, H, ρ, S and h. System (1.8), (1.10) exists in an involution and its general solution depends on a single arbitrary function of two variables.

New independent variables are introduced: the Lagrangian coordinate $\xi = \xi(t, r)$ and the modified time $\tau = \tau(t, r)$ in accordance with the equations

$$D_0\xi = 0, \quad \xi(0, r) = r, \quad kD_0\tau = 1, \quad \tau(0, r) = 0 \tag{1.11}$$

Then, $kD_0 = \partial_\tau$, and the solution of (1.10) with the condition $h(0, r) = 0$ has the form $h = \mathrm{tg}\,\tau$. The quantity

$$\eta = \cos\tau\sin\theta\cos\omega - \sin\tau\cos\theta \tag{1.12}$$

is defined and the quantity ζ, which is implicitly defined by the relation

$$\sqrt{1-\eta^2}\sin(\zeta + \varphi) = \cos\tau\cos\theta\cos\omega + \sin\tau\sin\theta \tag{1.13}$$

The general solution of system (1.8), expressed implicitly, has the form

$$F(\xi, \eta, \zeta) = 0 \tag{1.14}$$

with an arbitrary function F.

The behaviour of spherical trajectories of gas particles (the projections of trajectories onto the unit sphere) has been previously described [1] and it has been proved that any of them is a large circumference of a sphere and that the rate of translation of a particle along it, with respect to the time τ, is equal to unity. The value $\omega_0 = \pi/2$, which satisfies the conditions of uniqueness and definiteness of the solution on the whole sphere, is picked out among the initial data for the Cauchy problem.

The representation of the complete pattern of the motion of a special vortex is thereby reduced to determining the radial motion of the gas particles, which is described by system (1.7), (1.9), (1.10). This system has been integrated [1] in two cases: in the case of steady flows of an incompressible fluid (ρ = const) and the self-similar motions of a gas with an equation of stage $p = A\rho + B$ (A and B are constants) for which $U = 0$. The problem of the complete group analysis of a system which describes the radial motions of a gas has also been formulated. A description of a homogeneous special vortex in an ideal incompressible fluid is given in [4].

2. INITIAL DATA AND THE JACOBI MATRIX FOR A SPECIAL VORTEX

As a rule, invariant and partially invariant solutions have special initial data. For instance, the Jacobian matrix $J = \partial\mathbf{u}/\partial\mathbf{x}$ of the velocity field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ in barochronic motion has algebraic invariants at all instants of time which depend solely on the time [5] and, consequently, in the case of barochronic motions, the initial velocity field possesses the property that the algebraic invariants of the matrix $J_0 = \partial\mathbf{u}_0/\partial\mathbf{x}$ are real constants. It turns out that a similar property also holds in the case of a special vortex.

We will recall the formulae specifying a Jacobi matrix $J = (\nabla_i u^j)$ of the velocity vector field $\mathbf{u} = (U, V, W)$ in spherical coordinates (r, θ, φ) in terms of covariant derivatives [6]

$$\nabla_i u^j = \begin{Vmatrix} U_r & U_\theta - V & U_\varphi - W\sin\theta \\ r^{-1}V_r & r^{-1}(U_\theta + U) & r^{-1}(V_\varphi - W\cos\theta) \\ (r\sin\theta)^{-1}W_r & (r\sin\theta)^{-1}W_\theta & r^{-1}(W_\varphi/\sin\theta + U + V\mathrm{ctg}\theta) \end{Vmatrix} \quad (2.1)$$

The following assertion holds.

Theorem 1. The algebraic invariants of the Jacobi matrix (2.1) of the velocity field of the special vortex (1.2), (1.6) are functions of only the invariant independent variables t and r and are represented by the formulae

$$k_1 = r^{-2}(r^2U)_r - r^{-1}hH \quad (2.2)$$

$$k_2 = r^{-2}(rU^2)_r + r^{-1}HH_r - r^{-1}hHU_r - r^{-2}hUH \quad (2.3)$$

$$k_3 = r^{-2}(U - hH)(UU_r + HH_r) \quad (2.4)$$

Proof. We recall that an algebraic invariant k_i of order i of a matrix J is the sum of the principal minors of order i. So, for example, k_1 is the trace of the matrix and k_3 is the determinant of J (in the case of three-dimensional matrices).

We now calculate the first invariant for the solution (1.2), (1.6)

$$k_1 = \mathrm{div}\,\mathbf{u} = k_{10} + k_{11} \quad (2.5)$$
$$k_{10} = r^{-2}(r^2U)_r, \quad k_{11} = (r\sin\theta)^{-1}H((\cos\omega\sin\theta)_\theta + (\sin\omega)_\varphi)$$

In order to do this, we rewrite the equation of continuity in the form

$$D_0\ln\rho + k_{10} + k_{11} = 0, \quad D_0 = \partial_t + U\partial_r \quad (2.6)$$

The sum of the first two terms in equality (2.6) is equal to h/k by virtue of the second equality of (1.9) and, consequently, $k_{11} = -h/k$. On substituting this value of k_{11} into equality (2.5) and using the first equality of (1.9), we obtain formula (2.2).

Formulae (2.3) and (2.4) are proved by direct calculation but this is far more tedious. It essentially uses the second equation of the overdetermined system (1.8) for the extraneous function, that is, formulae (2.3) and (2.4) hold by virtue of the second equation of system (1.8).

Corollary. This initial data for a special vortex are special and the Jacobian matrix of the initial velocity field has algebraic invariants which depend solely on r.

In the case of regular partially invariant solutions of the type (1, 1) and (1, 2) of the equations of gas dynamics[1], the Jacobi matrix of the velocity field possesses the property formulated in Theorem 1. This enables us to propose the following hypothesis: for all regular partially invariant solutions, the Jacobi matrix of the vector velocity field has algebraic invariants which depend solely on the invariant independent variables.

Remarks. 1. This property of the matrix J picks out in an invariant manner the class of initial data of the corresponding solution.

2. The property is verified by explicitly calculating the invariants of the Jacobi matrix for all solutions.† It is interesting that it, unlike the formulae for the representation of the solution, has an invariant formulation.

A knowledge of the eigenvalues of the matrix J is also useful when analysing the solutions.

Lemma 1. The eigenvalues λ_i of the Jacobi matrix (2.1) of the velocity field of a special vortex (1.2), (1.6) are functions of only the invariant independent variables t and r and are represented by the following formulae

$$\begin{aligned} \lambda_1 &= r^{-1}(U-hH) \\ \lambda_{2,3} &= \frac{1}{2}(U_r+r^{-1}U\pm((U_r-r^{-1}U)^2-4r^{-1}HH_r)^{1/2}) \end{aligned} \tag{2.7}$$

In fact, substitution of expressions (2.7) into the representation of the invariants

$$k_1 = \lambda_1+\lambda_2+\lambda_3, \quad k_2 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3, \quad k_3 = \lambda_1\lambda_2\lambda_3$$

leads to formulae (2.2)–(2.4).

3. A STEADY-STATE SPECIAL VORTEX

System (1.7), (1.9), (1.10), which describes the radial gas motions, allows of a Lie algebra with the operator $T = \partial_t$. Solutions, which are invariant with respect to this algebra, can be constructed at once as the regular, partially invariant solutions of the equations of gas dynamics (1.3) with respect to a subalgebra with the basis T, X, Y, Z (see the paper cited in [1]. These solutions are of rank 1 and defect 1 and are represented in the form

$$U = U(r), \quad H = H(r), \quad \rho = \rho(r), \quad S = S(r), \quad \omega = \omega(r,\theta,\varphi) \tag{3.1}$$

Substituting these expressions into Eqs (1.7) and (1.10) we obtain the equations

$$\begin{aligned} &UU'+\rho^{-1}p' = r^{-1}H^2, \quad U(rH)' = 0, \quad US' = 0, \quad kUh' = h^2+1 \\ &k = r/H, \quad h = k[U(\ln\rho)'+r^{-2}(r^2U)'] \end{aligned} \tag{3.2}$$

where a prime denotes a derivative with respect to r.

It follows from system (3.2) that $U \neq 0$ (in the case when $U = 0$, the last equation of (3.2) leads to a contradiction). Consequently, the submodel describes the isentropic motions of a gas, $S = \text{const}$. It follows from the second equation of (3.2) that $rH = r_0H_0$, where $r_0, H_0 = \text{const}$. We put $a_0 = r_0H_0$. Then $k = r^2/a_0$ and the representation

$$U = a_0(h^2+1)/r^2h' \tag{3.3}$$

follows from the fourth equation of (3.2).

† Non-barochronic submodels of types (1, 2) and (1, 1) of the equation of gas dynamics. Preprint No. 1–99. Inst. Gidrodinamiki, Sib. Otd., Ross. Akad. Nauk, Novosibirsk, 1999.

Substituting expression (3.3) into the last formula of (3.2) for h and integrating, we obtain

$$\rho = R_0|h'|/\sqrt{1+h^2} \tag{3.4}$$

where the constant $R_0 > 0$. Consequently, all of the required functions are represented in terms of the function $h(r)$ and its derivatives. The pressure is determined from the equation of state. The function h is a solution of the first equation in (3.2) which is integrated once

$$\frac{1}{2}U^2 + I(\rho) + \frac{a_0^2}{2r^2} = b_0 \tag{3.5}$$

where $I(\rho) = \int dp/\rho$ is the enthalpy of the gas and the constant $b_0 > 0$.

Equation (3.5) is an invariant Bernoulli integral. In the case of a gas with a polytropic equation of state $p = \rho^\gamma$ ($\gamma > 1$ is the adiabatic exponent), relation (3.5) takes the form

$$|h'|^{\gamma+1} + \kappa\frac{a_0^2 - 2b_0r^2}{2r^2}(1+h^2)^{(\gamma-1)/2}h'^2 + m_0\frac{(1+h^2)^{(\gamma+3)/2}}{r^4} = 0 \tag{3.6}$$

$$\kappa = \gamma^{-1}(\gamma-1)R_0, \quad m_0 = (a_0^2\kappa)/2$$

The function $h = h(r)$, the solution of Eq. (3.6), determines the radial component of the velocity and the density in accordance with formulae (3.3) and (3.4). It also occurs in the integrals of ξ, η and ζ, which determine the spherical motion of the gas particles. Hence, the determination of a steady-state special vortex reduces to the integration of Eqs (3.6)

Put

$$r_* = a_0/\sqrt{2b_0} \tag{3.7}$$

It follows from the Bernoulli integral (3.5) that a steady-state special vortex is defined when $r \geqslant r_*$, rather than in the whole of space. As in the case of a gas source, a steady-state special vortex cannot be of a point character: the quantity r_* (3.7) determines the minimum radius of the sphere from which the steady-state special vortex "emerges". This property also holds in the case of a steady-state special vortex in an ideal incompressible fluid [1].

4. A HOMOGENEOUS SPECIAL VORTEX (A POLYTROPIC GAS)

We will now consider a gas with a polytropic equation of state $p = S\rho^\gamma$. The invariant subsystem (1.7), (1.9), (1.10) admits of a Lie algebra, which is specified by the extension operator

$$K = r\partial_r + U\partial_U + H\partial_H + \alpha\rho\partial_\rho + (\alpha+2)p\partial_p \tag{4.1}$$

with an arbitrary parameter $\alpha \in \mathbb{R}$. The variable t is an invariant of the operator K and, using it, it is possible to construct an invariant solution of rank 1, which has the representation

$$U = A(t)r, \quad H = C(t)r, \quad \rho = r^\alpha R(t), \quad p = r^{\alpha+2}P(t) \tag{4.2}$$

The representations for the entropy

$$S = r^m s(t), \quad s(t) = PR^{-\gamma}, \quad m = \alpha + 2 - \alpha\gamma \tag{4.3}$$

and the speed of sound

$$c^2 = \gamma p/\rho = \gamma r^2 B, \quad B = R^{-1}P \tag{4.4}$$

follow from the equation of state. The invariant functions (4.2)–(4.4) are homogeneous with respect to the variable r, which provides a basis for calling this solution a homogeneous special vortex.

The formulae

$$\rho^{-1}\nabla p = (\alpha+2)B\mathbf{x}, \quad \rho^{-1}p_r = (\alpha+2)Br \tag{4.5}$$

hold in the case of solution (4.2). In this case, all the functions of time A, C, R and P in (4.2), which determine the invariant part of the solution, are also expressed in terms of the function h and its derivatives (the special potential of the solution). The function $h(t)$ satisfies a third-order ordinary differential equation, the Schwarz equation with a definite right-hand side [7]. As before, the extraneous function ω is determined from the final relation of (1.14) after the function h has been found.

We will now obtain representations of all of the invariant functions (4.2) in terms of the functions h and its derivatives and derive an equation for h. In this section, a prime denotes a time derivative of the required functions.

On substituting the representations for H and U into the second equation of (1.7), we obtain

$$A = -C'/(2C) \tag{4.6}$$

For the function (4.2),

$$\rho^{-1}D_0\rho + r^{-2}(r^2U)_r = R^{-1}R' + (\alpha+3)A$$

Substituting this expression into formula (1.9) for h and taking account of the fact that $k = C^{-1}$, we obtain

$$h = C^{-1}[\ln RC^{-(\alpha+3)/2}]' \tag{4.7}$$

The representation

$$C = h'/(1+h^2) \tag{4.8}$$

follows from Eq. (1.10).

Substituting expression (4.8) into the equality (4.7), we obtain

$$R = R_0(1+h^2)^{-(\alpha+2)/2}(h')^{(\alpha+3)/2} \tag{4.9}$$

where R_0 is a constant of integration.

Substituting expression (2.2) for div $\mathbf{u}$ and c^2 from (4.4) into the equation for the speed of sound

$$D_0c^2 + (\gamma-1)c^2\operatorname{div}\mathbf{u} = 0$$

we obtain

$$B'/B - C'/C - (\gamma-1)[\ln(1+h^2)^{1/2}C^{3/2}]' = 0$$

Integrating this equation once, we obtain the representation

$$B = B_0(1+h^2)^{-\gamma}(h')^{(3\gamma-1)/2} \tag{4.10}$$

where $B_0 = \text{const}$. It is more convenient to describe the solution by specifying c^2 instead of p. Relations (4.4), (4.9) and (4.10) establish connections between these quantities in the given solution. We note that, from the physical meaning of the quantities ρ and c^2 and the representations (4.2), (4.4) and (4.9), (4.10), it follows that $R_0 > 0$, $B_0 > 0$.

Formulae (4.6), (4.8), (4.9) and (4.10) give representations of all the required functions (4.2) in terms of h. The required equation for h is obtained from the first equation of (1.7) which, after substituting representation (4.2), is transformed, by virtue of (4.5), into the Riccati equation for A

$$A' + A^2 + (\alpha+2)B = C^2 \tag{4.11}$$

Substituting the representations for A, B and C in terms of h into Eq. (4.11), we arrive at the required Schwarz equation. We will formulate the final result in the form of an assertion.

Theorem 2. The submodel of a special vortex, which is invariant with respect to the algebra (4.1), is given by the representation of the invariant functions (4.2) – (4.4) in which the functions A, B, C and R are expressed in terms of the auxiliary function h using the following formulae

$$A = -\frac{1}{2}\left(\ln\frac{|h'|}{1+h^2}\right)', \quad B = B_0(1+h^2)^{-\gamma}|h'|^{(3\gamma-1)/2}$$
$$C = (1+h^2)^{-1}h', \quad R = R_0(1+h^2)^{-(\alpha+2)/2}|h'|^{(\alpha+3)/2} \tag{4.12}$$

Where $\gamma > 1$ is the adiabatic exponent, $\alpha \in \mathbb{R}$ is the parameter of the submodel and $B_0 > 0$ and $R_0 > 0$ are constants of integration. The function $h(t)$ is the solution of the Schwarz equation

$$\{h\} \equiv \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = 2(\alpha+2)B_0\frac{|h'|^{(3\gamma-1)/2}}{(1+h^2)^{\gamma}} \tag{4.13}$$

The subsequent investigation of a homogeneous special vortex reduces to an analysis of the solution of Eq. (4.13). Unfortunately, there are only a few results on the representation and qualitative properties of its solution.

5. ALGEBRAIC INVARIANTS OF THE JACOBI MATRIX AND INITIAL DATA FOR A HOMOGENEOUS SPECIAL VORTEX

In the case of a homogeneous special vortex, Theorem 1 takes a more specific form.

Lemma 2. In the case of the homogeneous special vortex described in Theorem 2, the algebraic invariants and the eigenvalues of the Jacobi matrix are solely functions of time and are given by the formulae

$$k_1 = 3A - hC, \quad k_2 = 3A^2 - 2hAC + C^2$$
$$k_3 = A^3 - hA^2C + C^2A - hC^3 \tag{5.1}$$

$$\lambda_1 = A - hC, \quad \lambda_{2,3} = A \pm iC \tag{5.2}$$

Actually, substitution of the representation of the solution (4.2) into formulae (2.2) – (2.4) and (2.5) gives expressions (5.1) and (5.2).

Formulae (5.2), had using the function h, which defines the homogeneous special vortex, have the form

$$\lambda_1 = -\frac{1}{2}(\ln|h'|)', \quad \lambda_{2,3} = -\left(\frac{1}{2}\ln\frac{|h'|}{1+h^2} \pm i\,\mathrm{arctg}\,h\right)'$$

Remark. The initial data for a homogeneous special vortex are specified by special Jacobi matrices J_0 with constant invariants. The matrices J have invariants at all instants of time which depend solely on time. However, these matrices differ from the Jacobi matrices determining the barochronic solutions, since they satisfy different Riccati matrix equations.

The search for a solution of the form of (4.2) reduces to the solution of the Schwarz equation (4.13). We will now show that the initial data for it,

$$h_0 = h(0), \quad h_0' = h'(0), \quad h_0'' = h''(0) \tag{5.3}$$

which are arbitrary constants, are expressed in terms of the initial physical data

$$\mathbf{u}_0 = \mathbf{u}(0, \mathbf{x}), \quad \rho_0 = \rho(0, \mathbf{x}), \quad c_0 = c(0, \mathbf{x})$$

of solution (4.2).

Expressions for the initial physical data of the solution

$$\begin{aligned} U_0 &= A(0)r, \quad V_0 = C(0)r\cos\omega_0, \quad W_0 = C(0)\sin\omega_0 \\ \rho_0 &= r^{\alpha}R(0), \quad c_0^2 = \gamma r^2 B(0) \end{aligned} \tag{5.4}$$

with the constants $A(0)$, $C(0)$, $R(0)$, $B(0)$, follow from formulae (1.2) and (4.2). The function $\omega_0 = \omega(0, r, \theta, \varphi)$ gives the initial data for system (1.8).

We use formulae (4.12) in order to connect the initial data (5.3) and the numbers $A(0)$, $B(0)$, $C(0)$, $R(0)$ in (5.4). It is more convenient to use the representation $h = \operatorname{tg}\tau$, where $\tau = \tau(t)$ is the modified time (1.11). The derivatives h' and h'' are then uniquely expressed in terms of the function τ and its derivatives and the initial data (5.3) are recalculated in terms of $\tau_0 = \tau(0)$, $\tau_0' = \tau'(0)$, $\tau_0'' = \tau''(0)$. Equation (4.13) can be rewritten for the function τ. As before, it will be a solution of the Schwarz equation but with a modified right-hand side. Specific formulae are not very important here but it should be emphasized that the initial data (5.3) are expressed in terms of the initial physical data (5.4) for a homogeneous special vortex

$$\cos\tau_0 = \left(\frac{R(0)}{R_0}\right)^{\gamma}\frac{B(0)}{B_0}C^{(\alpha\gamma+1)/2}(0), \quad \tau_0' = C(0), \quad \tau_0'' = -2A(0)C(0)$$

6. INTEGRALS OF THE SCHWARZ EQUATION WHEN $\gamma = 5/3$

The case of an exclusive value of the adiabatic exponent is distinguished from the point of view of the integration of the Schwarz equation, which takes the form

$$\{h\} = 2(\alpha+2)B_0 h'^2/(1+h^2)^{5/3} \tag{6.1}$$

Lemma 3. The function $h = h(t)$, which is a solution of the Schwarz equation

$$\{h\}_t = \Phi(h)h_t'^2 \tag{6.2}$$

($\Phi(h)$ is a given function), is related by the equation

$$h_t'(C_1 + C_2 t)^2 = Q^2(h) \tag{6.3}$$

(C_1 and C_2 are constant) to the solution $Q = Q(h)$ of the linear equation

$$\frac{d^2Q}{dh^2} - \frac{1}{2}\Phi(h)Q = 0 \tag{6.4}$$

Proof. By well-known results in [7, 8], we have the following chain of equalities. Equation (6.3), after changing the roles of the variables, takes the form

$$\{t\}_h = -\Phi(h) \tag{6.5}$$

Suppose $t_h' > 0$. Then, the general solution of Eq. (6.4) is represented by the formula

$$Q = \frac{1}{\sqrt{t_h'}}(C_1 + C_2 t(h)) \tag{6.6}$$

where $t = t(h)$ is the general solution of Eq. (6.5) and C_1, C_2 are arbitrary constants. On transforming the relation $t = t(h)$ and squaring both sides of equality (6.6), we arrive at relation (6.3)

Corollary. The critical points t_* of the function $h = h(t)$ give the zeros of the function $Q = Q(h)$ and the values of h for which $h'(t_*) = 0$ make Q vanish: $Q(h) = 0$.

The proof follows from relation (6.3).

7. A QUALITATIVE DESCRIPTION OF THE RADIAL MOTION OF A GAS WHEN $\gamma = 5/3$

Reduction of the Schwarz equation (6.1) to Eqs (6.3) and (6.4) enables us to describe the radial motion of a gas. Integrating the equation of the trajectories $dr/dt = U$ in the case of the function U (4.2), we obtain

$$r^2(1+h^2)^{-1}h' = r_0^2 C_0 \tag{7.1}$$

where r_0, $C_0 = C(0)$ are constants of integration. Formula (7.1) determines the Lagrange coordinate $\xi = r|C|^{1/2}$ (1.11) for the given solution. Equation (7.1) yields the contact characteristic for the given solution, that is, it determines the surface which is "woven" from the trajectories. It is a surface of rotation about the Ot axis in the space of events $\mathbb{R}^4(t, \mathbf{x})$ which envelopes a family of spheres with centres at points of the Ot axis and radii given by Eq. (7.1): $r^2(t) = r_0^2 C_0/C(t)$.

We will now calculate the values of the density and the speed of sound in a particle, that is, when $r = r(t)$ which is determined by Eq. (7.1). We obtain

$$\rho = r_0 R_0 |C_0|^{3/5} \frac{|h'|^{3/2}}{1+h^2}, \quad c^2 = \frac{5}{3} r_0^2 C_0 \frac{|h'|}{(1+h^2)^{3/2}} \tag{7.2}$$

It follows from Eq. (7.1) that the pattern of the gas motion is determined by the number of critical points of the function $h(t)$. Actually, according to Eq. (7.1), $r \to +\infty$ in the case of a differentiable function h when $t \to t_*$, where t_* is the critical point of the function h at which $h'(t_*) = 0$.

It follows from formulae (7.2) that a critical point of the function h is a vacuum point: $\rho(t_*) = c(t_*) = 0$. According to integral (6.3), the critical points of the function h coincide with the zeros of the function $Q(h)$, which is a solution of Eq. (6.4). It follows from general theory [9] that the number of zeros of the solution of second-order linear equation (6.4) depends on the sign of the coefficient Φ. For Eq. (6.1),

$$\Phi = 2(\alpha+2)B_0(1+h^2)^{-5/3}$$

Consequently, sign Φ = sign $(\alpha + 2)$, since $B_0 > 0$, by virtue of relations (4.4) and (4.12).

We will now examine possible cases.

A. Suppose $\Phi > 0$ that is, $\alpha + 2 > 0$. Each non-zero solution of Eq. (6.4) then has no more than a single zero and, consequently, it is non-zero for all sufficiently large values of h.

In this case, the function $h(t)$ has no more than a single critical point $h'(t_*) = 0$ when $t = t_*$ such that $Q(h_*) = 0$ for $h_* = h(t_*)$. The qualitative behaviour of the functions $h = h(t)$ and $Q = Q(t)$ in this case is shown in Fig. 2. It follows from relations (7.1) and (7.2) that $r \to +\infty$, $c \to 0$, $\rho \to 0$ when $t \to t_*$. The solution is defined in the intervals $(-\infty, t_*)$ and $(t_*, +\infty)$. Gas particles depart to infinity when $t \to t_* \pm 0$ and, at the same time, the density of the gas cloud decreases, tending to the limiting state of a vacuum (Fig. 3).

B. Suppose $\Phi < 0$. that is, $\alpha + 2 < 0$. Each non-trivial solution $Q(h)$ of Eq. (6.6) as well as its derivative then has infinitely many zeros, that is, each solution contains an infinite set of oscillations while, at the same time, the distances between neighbouring zeros remain bounded.

Suppose $h_i = h(t_i)$, $i \in N$ are the zeros of the solution: $Q(h_i) = 0$. Then, at the critical points t_i, we have $h'(t_i) = 0$ (Fig. 4). According to equality (7.1), the gas particles depart to infinity: $r \to +\infty$ when $t \to t_i$. At the same time, $c(t_i) \to 0$, $\rho(t_i) \to 0$. The gas motion is determined in each of the intervals (t_i, t_{i+1}) and is described in the following manner. A rarefied gas cloud ($t = t_i$), on condensing, approaches an observer at a minimum distance $r_{\min}$ and then moves away from him, dispersing to the limiting state of a vacuum at infinity when $t = t_{i+1}$ (Fig. 5). The contact characteristics, that is, the surfaces in $\mathbb{R}^4(t, \mathbf{x})$, completely consisting of the trajectories, are surfaces of revolution about the Ot axis, the generatrices of which are shown in Figs 3 and 5.

A complete description of the gas motion is possible when Eq. (1.4), which specifies the motion of the particles on the spheres r = const, is taken into account, and this motion is added to the radial motion. It has been shown in [1] that, in a special vortex, each gas particle during its motion does not leave a plane, the position of which in the space $\mathbb{R}^4(t, \mathbf{x})$ depends on the initial data: the position and velocity of the gas particle. On the whole, these solutions describe the motion of a gas cloud which is formed at sufficiently large r from a rarefied gaseous medium. The types of motion A and B which have been described above are distinguished by their times of existence: infinite in case A, since the solution is defined in the interval $(-\infty, t_*)$ or $(t_*, +\infty)$, but finite (t_i, t_{i+1}) in case B.

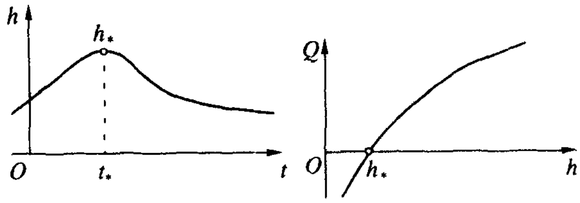

Fig. 2

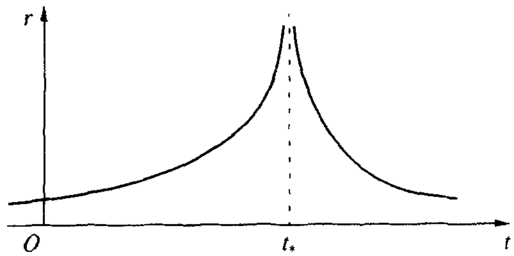

Fig. 3

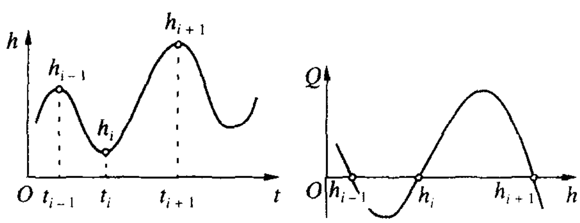

Fig. 4

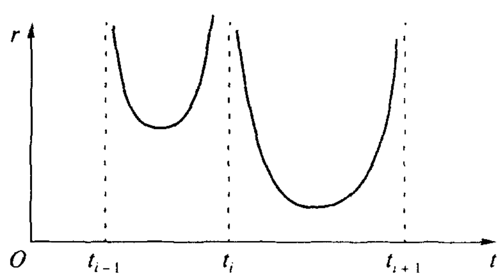

Fig. 5

A more detailed description of the motion is associated with the existence of simple particular solutions of the Schwarz equation and an analysis of the solutions of Eq. (1.14) in the case of specific functions F.

8. THE BAROCHRONIC HOMOGENEOUS SPECIAL VORTEX

The Schwarz equation (4.13), when $\alpha = -2$, and for any γ, has the simple general solution

$$h = (at+b)/(ct+d) \tag{8.1}$$

in which the constants of integration a, b, c and d are such that $\Delta = ad - bc \neq 0$ are related by a single condition. We shall determine it later so that the formulae representing the solution have the simplest form. The function (8.1) defines a non-isentropic, barochronic solution of the equations of gas dynamics since it follows from relations (4.2) and (4.3) that

$$p = P(t), \quad \rho = r^{-2}R(t), \quad S = r^{4\gamma}PR^{-\gamma} \tag{8.2}$$

Solution (8.2) can be called a barochronic, homogeneous special vortex. The mathematical foundations of the theory of the barochronic motions of a gas have been described earlier [5]. The barochronic motion of a gas has the following characteristic features.†

1. The gas particles move along rectilinear trajectories, the velocity of the particles is constant along the trajectories, but is different for different gas particles. The special initial data which guarantees such a motion is referred to in Section 2.

2. The trajectory mapping which compares the initial position of each particle with its position at an instant of time $t > t_0$ degenerates at a certain finite instant of time $t = t_c$. This corresponds to a collapse of the density: $\rho \to +\infty$ when $t \to t_c$ in the manifold Σ_c, the dimension of which is less than the dimension of the motion. For instance, during the barochronic motion of a bounded gas volume it collapses when $t = t_c$ into a part of a surface, a curve or a point depending on the degree of degeneracy of the trajectory mapping. Physically, the motion can be treated as an ultrastrong compression of a gas by a piston of special configuration formed by the contact characteristics, the pressure on which changes with time according to a specified law.

Suppose the condition imposed the constants a, b c and d, about which we have spoken above, has the form $ab + cd = 0$. The numbers a, c and σ_0, such that

$$b = -\sigma_0 c, \quad d = \sigma_0 a, \quad \Delta = \sigma_0(a^2 + c^2)$$

can be chosen as the three essential constants on which the solution depends. Formulae (4.2), which give the solution in the case of a function h of the form of (8.1), take the form

$$U = rt/(t^2 + \sigma_0^2), \quad \rho = |\Delta|^{1/2}/(r^2|ct + d|) \tag{8.3}$$

The equations of the radial motion of the gas particles $dr/dt = U$ are integrated:

$$(r/r_0)^2 - (t/\sigma_0)^2 = 1 \tag{8.4}$$

The motion of the gas particles in the space $\mathbb{R}^4$ $(t, \mathbf{x})$ takes place along the rectilinear generatrices of the unparted hyperboloid of rotation, which is given by Eq. (8.4). It is the enveloping surface of a family of two-dimensional spheres with centres at points on the Ot axis and radii

$$r = r_0\sqrt{1 + \sigma_0^{-2}t^2} \tag{8.5}$$

There are two families of rectilinear generatrices for a unparted hyperboloid. The gas particles move along the generatrices of the first family in the direction of the increase in time into the future and, along the lines of the second family, in the direction of the increase in time into the past. The equations of gas dynamics are invariant under time reversal: $t \to -t$, $\mathbf{u} \to -\mathbf{u}$. This involution changes the places of the generatrices of the two families. The initial data for each gas particle are specified by Lagrange coordinates, that is, the position of the particle on the hyperboloid and the velocity which is kept constant along the trajectories of the particles. The complete motion of the gas is made up of the radial motion described above and a spherical motion described by integral (1.14). The motion of a gas, occupying, when $t = 0$, a sphere of radius r_0, can serve as an illustration. In all of the following instants of time, the gas, while expanding according to the law (8.3), also occupies a sphere of radius (8.5) (Fig. 6).

In order to describe the collapse in such a motion, which sets in when $t \to t_c = -d/c$, it is necessary to include Eq. (1.14) for the spherical motion. We will consider the special case of the motion of a gas which is independent of the length φ. Equation (1.14) for the function ω then takes the form

$$\sin\theta\cos\omega = h\cos\theta + \sqrt{1 + h^2}F(\xi) \tag{8.6}$$

It follows from this equation that the function F of the Lagrangian variable ξ, which is determined by the initial gas distribution, cannot be arbitrary and only obeys the condition $|F| \leqslant 1$. Actually, the relation

†CHUPAKHIN, A. P., Barochronics motions of a gas. General properties and submodels of types (1, 2) and (1, 1). Preprint No. 4–98. Inst. Gidrodinamiki, Sib. Otd., Ross. Akad. Nauk, Novosibirsk, 1998.

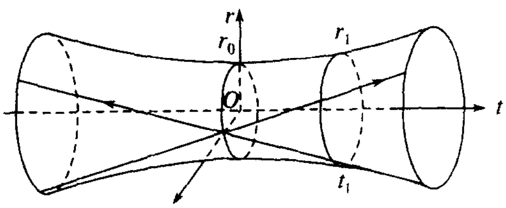

Fig. 6

$$\sin(\theta - \psi) = \sqrt{\frac{1 + h^2}{\cos^2\omega + h^2}}F, \quad \psi = \mathrm{arctg}\frac{h}{\cos\omega}$$

from which the constraint on F mentioned above follows, is a consequence of Eq. (8.6).

Substituting function (8.1) into equality (8.6), we obtain

$$\begin{aligned} |ct + d|\sin\theta\cos\omega &= \varepsilon(at + b)\cos\theta + \sqrt{a^2 + c^2}(t^2 + \sigma_0^2)^{1/2}F(\xi) \\ \varepsilon &= \mathrm{sign}(ct + d) \end{aligned} \qquad (8.7)$$

Equation (8.7) specifies the function $\omega = \omega(t, r, \theta)$ at any instant of time $t \neq t_c$. When $t = t_c$, its left-hand side vanishes and it only relates the independent variables, defining one more equation of the collapse manifold. Consequently, the collapse manifold is a one-dimensional curve and is given in the space of events $\mathbb{R}^4$ $(t, \mathbf{x})$ by the equation.

$$\Sigma_c\colon t = t_c, \quad r = r_0\sqrt{1 + \sigma_0^{-2}t_c^2}, \quad \cos\theta = \varepsilon F(\xi) \qquad (8.8)$$

The manifold Σ_c lies in a two-dimensional sphere and, in the general case, is part of an arc $\theta = \theta_c$ which follows from expression (7.1) or the Lagrangian variable: $\xi = r^2C$. Its form depends on the function F.

9. CONCLUSION

The qualitative analysis of the invariant submodels of a special vortex which has been carried out shows that, in these submodels, the corresponding motion of the gas can be studied more completely than in the general case.

1. The equations of the radial motion of a gas reduce, in the case of these submodels, to a single ordinary differential equation which is specific for each of them. All of the invariant functions are represented in terms of the solution of this equation, that is, the function h and its derivatives. For a steady-state special vortex, this equation is of the first order, but is not self-similar and is not integrated in quadratures. An investigation of the qualitative properties of this solution is a separate problem. It has been proved that a steady-state special vortex is defined when $r \geqslant r_* > 0$, rather than in the whole space.

In the case of a homogeneous special vortex, the equation for h is the Schwarz equation with a right-hand side which is rational with respect to h and h'. For any adiabatic exponent $\gamma = 5/3$, the equation "decomposes" into two equations: a non-linear first-order equation and a linear second-order equation, the solution of which gives the right-hand side of the first equation. Oscillating and non-oscillating solutions of the second equation have been investigated. The motions of a gas cloud correspond to these solution. This cloud is formed at a considerable distance from an observer from a rarefied medium and approaches him at a certain minimum distance after which the stage of withdrawal and rarefaction of the cloud begins. Finite or infinite time intervals of the existence of the solution correspond to different forms of the solution.

A barochronic homogeneous special vortex has been completely described.

2. It has been proved that a special vortex possesses an interesting property. The algebraic invariants of the Jacobi matrix J of the velocity field of this solution depends solely on the invariant independent variables. Representations for them and for the eigenvalues of J have been found. In the case of the matrix J_0 of the initial velocity field, the invariants depend solely on r. The hypothesis has been formulated

that a similar property holds for all regular partially invariant solutions. The description of vector fields of such a form is interesting. This has only been done in the case of barochronic solutions for which the matrix J_0 has constant invariants [5].

3. To describe the submodels of a special vortex, the two-sphere algorithm for constructing the exact solutions is equivalent to the single-sphere model. Steady-state and homogeneous special vortices can be constructed as partially invariant submodels with respect to four-dimensional subalgebras of the algebra of the symmetry of the equations of gas dynamics after one step and, also, as invariant submodels of a special vortex, after two steps. In the case of invariant solutions, the conditions for the single-step and multistep algorithms to be equivalent were established by Ovsyannikov [10] (the LOT lemma). These conditions are still unknown in the case of partially invariant solutions. Preliminary considerations show that they will not be very different from the conditions of the LOT lemma. For instance, in the case of the L_{13} algebra, which is allowed by the equations of gas dynamics in the case of a polytropic gas with an arbitrary adiabatic exponent γ, the invariant subsystem of the special vortex admits of an algebra which is a factor algebra of the normalizer of the algebra $L_3 = \langle X, Y, Z\rangle$ in L_{13}. This algebra has the basis of operators

$$\partial_t, \quad t\partial_t + \mathbf{x}\partial_\mathbf{x}, \quad t\partial_t - \mathbf{u}\partial_\mathbf{u} - 2p\partial_p, \quad \rho\partial_\rho + p\partial_p$$

A steady-state special vortex is generated by the algebra $\langle\partial_t\rangle$, and a homogeneous special vortex is generated by the operator (4.1), which is a linear combination of three dilatation operators from L_4.

The complete list of invariant and partially invariant submodels of a special vortex can be obtained when investigating the group property of the equation from Section 1. A fraction of these submodels is contained in the optimal systems of subalgebras of the algebra of the symmetry of the equations of gas dynamics for a polytropic gas with an arbitrary adiatatic exponent γ and with the exclusive value $\gamma = 5/3$.† The investigation of these submodels is a challenging problem.

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